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Towards a theory of the surface conductance coefficient in solids.
Part 1

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Abstract

The best governing equations for the steady state and cooling in a cylindrical aluminium bar have been determined with precision. As a first stage, a single exponential law was not found to be a good approximation for a direct calculation of the surface conductance coefficient, contrary to what had previously been found for iron. A more general theory to explain the dependence of the surface conductance coefficient on time position, and temperature could be based, in principle, on the use of the hypergeometric series. It would probably need a great deal of experimental data from several solids, both good and poor conductors.

INTRODUCTION

The surface conductance coefficient H, also called the convective heat transfer coefficient [1], is one of the more difficult thermophysical parameters to calculate because of its dependence on other variables such as the nature, shape, and conductivity of the material, and the temperature gradient set up in it. However, some easy approximations can be applied which enable the order of magnitude of H in a practical problem to be estimated. An example is Newton's law of cooling: the heat flux across a surface is proportional to the temperature difference $(T - T_0)$ between the surface and the surrounding medium. Assuming Newton's law, the one-dimensional differential equation for the heat flow in a bar with no internal source of heat is given by [1, 2]

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \frac{Hp}{c\rho w}(T - T_0) \tag{1}$$

where $\kappa = K/\rho c$ is the thermal diffusivity, K is the thermal conductivity, ρ is the density, c is the specific heat, p is the perimeter, and w is the cross-section of the bar.

For steady state cases $(\partial T/\partial t = 0)$ and for cooling $(\partial^2 T/\partial x^2 = 0)$ in the bar, one can readily obtain the value of H.

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In the first case, the equation to be solved reduces to

$$\frac{\mathrm{d}^2T}{\mathrm{d}x^2}-\frac{Hp}{Kw}T=0 \tag{2}$$

where T is the temperature excess over the surroundings, i.e. the temperature of the medium T_0 is taken as zero. The solution of eqn. (2) is an exponential law

$$T_i = T_1 \exp(-mx) \tag{3a}$$

where

$$m^2=\frac{Hp}{Kw} \tag{3b}$$

In the second case, the equation to be solved is

$$\frac{\mathrm{d}T}{\mathrm{d}t}-\frac{Hp}{c\rho w}T=0 \tag{4}$$

whose solution is

$$T' = T_i \exp(-m't) \tag{5a}$$

where

$$m'=\frac{Hp}{c\rho w} \tag{5b}$$

In eqns. (3a) and (5a), T_1 is the temperature of the first point in the bar and T' denotes the cooling of a point i of the bar which started from the steady state at temperature T_i. Thus, because m and m' can be determined experimentally, H can be obtained from either eqn. (3b) or from eqn. (5b). If it is supposed that the surface conductance is the same in both the steady state and in cooling, then the relationship

$$m' = \kappa m^2 \tag{6}$$

must always be satisfied.

This theory is not always supported by experimental results. There is a strong dependence on the nature of the medium, as was shown in earlier studies [3, 4], in which eqn. (6) was found to be valid for a good conducting material (iron [3]) but not for a poor conductor (a plastic [4]). A broader theory for the surface conductance has to be constructed in three stages: first, measuring steady states and cooling for a wide range of temperatures and materials; secondly, finding the best governing equation for the experimental results; finally identifying the coefficients obtained in the best fit with the unknown thermodynamic properties of interest, such as the surface conductance coefficient H. The final step is the most problematical, because, among other difficulties, it could require the modification of the

TABLE 1

Physical properties and geometrical characteristics of the aluminium bar at 20°C

Parameter[a]	Value	Parameter[a]	Value
ρ (kg m^{-3})	2.7×10^3	L (m)	0.53
c (J kg^{-1} K^{-1})	8.96×10^2	d (m)	6×10^{-2}
K[b] (J s^{-1} m^{-1} K^{-1})	2.3×10^2	p (m)	9.4×10^{-2}
κ (m^2 s^{-1})	9.5×10^{-5}	w (m^2)	7.1×10^{-4}

[a] For definition of terms, see text. [b] At 100°C.

"raw" linear Newton's law of cooling to some more complex behaviour, leading to a different eqn. (1) for the heat flow.

The present study is devoted to following the three steps outlined above in order to build up a theory, if possible, of the surface conductance coefficients in solids. A good conducting material with wide use in industry — aluminium — has been chosen as the starting point of this work.

EXPERIMENTAL DETAILS AND RESULTS

Table 1 lists the physical properties ρ, c, K and κ of aluminium [5], and the geometrical characteristic L (length), d (diameter), p and w of the bar (in SI units). The basic experimental procedure was as described in ref. 6 for an iron bar, but adapted to the present case. Briefly, the temperature was measured along the bar by thermocouples set in ten holes drilled perpendicularly from a generatrix to the axis. The thermocouples were connected to a digital thermometer. The distances (x in cm) between thermocouples are listed in the upper row in Table 2; the first thermocouple

TABLE 2

The temperature excess distributions (°C) corresponding to the ten steady states studied

Steady state case	Distance x (cm)									
	0	4	8	12	16	20	24	28	32	36
I	73.5	64.7	57.0	50.9	45.0	40.4	36.2	33.1	31.1	28.7
II	66.8	59.0	52.2	46.7	41.5	37.4	33.7	30.9	29.0	26.9
III	57.2	50.5	44.5	39.8	35.2	31.7	28.5	26.1	24.5	22.6
IV	52.6	46.6	41.4	37.3	33.3	30.2	27.3	25.2	23.8	22.2
V	46.5	41.1	36.4	32.7	29.1	26.4	23.7	21.8	20.6	19.1
VI	41.4	36.8	32.6	29.3	26.1	23.5	21.2	19.5	18.3	17.1
VII	35.9	31.7	28.0	25.1	22.2	20.0	17.9	16.4	15.4	14.3
VIII	30.7	27.2	24.1	21.7	19.2	17.5	15.7	14.5	13.6	12.6
IX	25.0	22.2	19.7	17.8	15.9	14.4	13.0	12.0	11.4	10.6
X	20.4	18.2	16.3	14.7	13.2	12.0	11.0	10.2	9.6	9.0

was taken as the coordinate origin and located 5 cm away from the end of the bar, where a resistive coil connected to an a.c. variable transformer heated the bar. The bar was carefully isolated with asbestos cord, and the room temperature was measured with a thermometer inserted in a copper block away from the heat source.

Steady state

Table 2 lists the temperature at every point of the bar in the steady state for the ten cases studied. These cover a wide range of temperature excess over the surroundings of 53.1°C (from 73.5°C in case I to 20.4°C in case X). The ten cases were selected to give a temperature difference between consecutive cases of around 5–7°C at the first thermocouple.

Following the prediction of the theory for the steady state described in the Introduction, the experimental data should satisfy a single exponential law such as eqn. (3a). Table 3 lists the results of this fit with the standard deviation σ for T_1, m and H, from eqns. (3a) and (3b). The last column is the sum of the squares of the differences between the experimental and theoretical values: $sqdif = \sum_{i}^{N} (T_{i,e} - T_{i,t})^2$. This is a measure of the goodness of the fit to an a priori law [7]. It can be seen that the fit is fairly good, the poorest results being for the two highest temperatures (cases I and II). The theoretical T_1 values in Table 3 are slightly below the experimental T_1 values given in the first column in Table 2, the difference increasing with

TABLE 3

Parameters for the fit of the experimental steady state data in Table 2 to an exponential law (eqn. (3a))

Steady state case	Parameter [a]			*sqdif*
	$T_1 \pm \sigma$	$-m \pm \sigma$	$H \pm \sigma$ [b]	
I	72.0 ± 0.7	2.77 ± 0.06	13.3 ± 0.3	13.59
II	65.4 ± 0.7	2.68 ± 0.07	12.5 ± 0.3	11.19
III	56.0 ± 0.7	2.73 ± 0.08	12.9 ± 0.4	8.19
IV	51.4 ± 0.7	2.54 ± 0.09	11.2 ± 0.4	7.76
V	45.5 ± 0.7	2.6 ± 0.1	11.7 ± 0.5	5.80
VI	40.6 ± 0.7	2.6 ± 0.1	11.7 ± 0.5	4.11
VII	35.2 ± 0.7	2.7 ± 0.1	12.7 ± 0.5	3.09
VIII	30.1 ± 0.7	2.6 ± 0.1	11.7 ± 0.5	2.31
IX	24.4 ± 0.7	2.5 ± 0.2	10.9 ± 0.9	1.75
X	20.0 ± 0.7	2.4 ± 0.2	10.0 ± 0.8	1.13
		2.6 ± 0.1 [c]	11.9 ± 1.0 [c]	

[a] For definition of parameters, see text. [b] Eqn. (3b). [c] Mean value.

TABLE 4

Parameters for the fit of experimental data for the cooling of the first point of the bar to an exponential law (eqn. (5a)) for four selected cases

Case selected	Parameter[a]			*sqdif*
	$T_1' \pm \sigma$	$-m' \pm \sigma$	$H \pm \sigma$[b]	
I	78.1 ± 0.7	0.479 ± 0.006	8.8 ± 0.1	33.52
III	62.3 ± 0.7	0.486 ± 0.008	8.9 ± 0.1	19.02
VI	43.6 ± 0.7	0.44 ± 0.01	8.0 ± 0.2	9.01
VIII	33.2 ± 0.7	0.46 ± 0.01	8.4 ± 0.2	1.76
		0.47 ± 0.02[c]	8.6 ± 0.4[c]	

[a] For definition of parameters, see text. [b] Eqn. (5b). [c] Mean values.

temperature. The *m* values obtained show an upward trend with temperature, as do the *H* values calculated from eqn. (3b). There mean values are given in the last row of Table 3.

Cooling

Four of the ten cases studied in the steady state were chosen for the cooling study. Their temperature differences at the first point of the bar were about 10–20°C between consecutive cases. Table 4 lists the cases selected and the values obtained for the fit to an exponential law (eqn. (5a)). It can be seen that the cooling results behave quite differently from the steady state results: the experimental T' values in Table 4 are considerably higher than the corresponding entries in the first column in Table 2; the m' values fluctuate with no tendency to increase or decrease with temperature, as is also the case for H calculated from eqn. (5b); the *sqdif* is up to nearly three times greater (case I) than in the steady state. The fit of cooling to an exponential law is thus worse than for the steady state. The last row gives the mean values of m' and H. Comparing this value of H (8.6) with that obtained in the steady state (11.9), one must conclude that either eqn. (6) is no longer satisfied because the H' values are different, or eqn. (6) is satisfied within a large statistical error of about 30%. The first possibility implies that if the single exponential law is sufficiently accurate in describing the steady state and cooling, then two conductance coefficients H and H' must be defined for the same material, depending on the temperature regime established in the solid (whether steady state or cooling). This possibility does not seem physically convincing. The second possibility allows there to be only one H, independent of the temperature regime, but a better fit to the steady state and cooling must be found.

Seeking other solutions

The next step is to attempt to fit the experimental data to a more accurate analytical expression, bringing the value of *sqdif* as close to zero as possible. The easiest choice is a double exponential, which was found to be the best governing equation for the steady state and cooling in iron and plastic bars [6, 8], described by

$$T_i = a_1 \exp(-m_1 x) + a_2 \exp(-m_2 x) \tag{7a}$$

$$T' = a'_1 \exp(-m'_1 t) + a'_2 \exp(-m'_2 t) \tag{7b}$$

for the steady state and cooling respectively. The constants have dimensions of temperature and, for the initial values of $x = t = 0$, have to satisfy

$$T_1 = a_1 + a_2 = T' = a'_1 + a'_2 \tag{8}$$

Table 5 lists the results of fitting the steady states of Table 2 to eqn. (7a). Clearly the exponent m_2 of the second exponential is practically zero, eqn. (7a) being thus reduced to a single exponential plus a constant a_2. The remaining exponent m_1 is practically constant within the range of

TABLE 5

Parameters for the fit of the experimental steady state data in Table 2 to a double exponential law (eqn. (7a))

Steady state case	Parameter [a]				*sqdif*
	$a_1 \pm \sigma$	$-m_1 \pm \sigma$	$a_2 \pm \sigma$	$-m_2 \pm \sigma$	
I	56.9 ± 0.1	4.4 ± 0.1	16.8 ± 0.6	$<10^{-5}$	0.59
II	50.9 ± 0.1	4.3 ± 0.1	16.1 ± 0.6	$<10^{-5}$	0.36
III	44.0 ± 0.2	4.4 ± 0.2	13.4 ± 0.6	$<10^{-5}$	0.31
IV	38.7 ± 0.2	4.4 ± 0.2	14.0 ± 0.6	$<10^{-5}$	0.25
V	34.9 ± 0.2	4.4 ± 0.2	11.7 ± 0.6	$<10^{-5}$	0.23
VI	31.5 ± 0.3	4.2 ± 0.3	10.0 ± 0.7	$<10^{-5}$	0.23
VII	27.7 ± 0.4	4.3 ± 0.3	8.3 ± 0.7	$<10^{-5}$	0.17
VIII	23.2 ± 0.4	4.3 ± 0.3	7.6 ± 0.6	$<10^{-5}$	0.10
IX	18.4 ± 0.5	4.3 ± 0.3	6.7 ± 0.7	$<10^{-5}$	0.09
X	14.7 ± 0.8	4.3 ± 0.4	5.8 ± 0.4	$<10^{-5}$	0.03

[a] For definition of parameters, see text.

TABLE 6

Parameters for the fit of experimental data for the cooling of the first point of the bar to a double exponential law (eqn. (7b))

Case selected	Parameter[a]				*sqdif*
	$a'_1 \pm \sigma$	$-m'_1 \pm \sigma$	$a'_2 \pm \sigma$	$-m'_2 \pm \sigma$	
I	73.85 ± 0.01	0.556 ± 0.006	6.4 ± 0.2	0.136 ± 0.005	4.16
III	61.0 ± 0.7	0.544 ± 0.009	2.839 ± 0.006	0.072 ± 0.006	4.57
VI	30.55 ± 0.01	0.607 ± 0.009	14.5 ± 1.0	0.28 ± 0.03	1.30
VIII	33.0 ± 0.7	0.48 ± 0.02	0.563 ± 0.009	0.01 ± 0.01	0.80

[a] For definition of parameters, see text.

temperature studied. Because *sqdif* is very small for all the cases studied, the sum of a_1 and a_2 gives the experimental value T_1 in the first column of Table 2 very accurately, as predicted by the left-most equality in eqn. (8).

The results for fitting the cooling to a double exponential (eqn. (7b)) are shown in Table 6. The *sqdif* values are not as good as in the steady state (Table 5), but much better than for the cooling using the single exponential (Table 4). However, the sum of the coefficients a'_1 and a'_2 is greater than T' in Table 4 and thus much greater than the experimental values T_1 in the first column of Table 2. Therefore, for the cooling, eqn. (8) becomes $T_1 < T' < a'_1 + a'_2$. Both $a'_1 > a'_2$ and $m'_1 > m'_2$, and given the magnitudes of these coefficients, the second exponential can be considered as a “correction” to the single exponential law.

The best fit for the steady state and cooling do not verify the single exponential law defined by eqns. (3a) and (5a), so it is not possible to deduce the surface conductance from eqns. (3b) and (5b).

A different approach

A way to unify both sets of experimental result, i.e. a single exponential plus a constant for the steady state, and a double exponential (with one small exponential being a correction to the other) for the cooling; is to consider a more general *n*th order polynomial approximation to the data

$$T = a + bx + cx^2 + dx^3 + \dots \tag{9a}$$

$$T = a' + b't + c't^2 + d't^3 + \dots \tag{9b}$$

for the steady state and cooling, respectively, where $a, b, \dots d'$ are constants to be determined by the experimental data. The fits for the two

TABLE 7

Parameters for the fit of the experimental steady state data in Table 2 to a second order polynomial law (eqn. (9a))

Steady state case	Parameter[a]			*sqdif*
	a	$-b$	*c*	
I	73.14	2.17	0.026	0.81
II	66.48	1.93	0.023	0.56
III	56.94	1.67	0.020	0.51
IV	52.33	1.47	0.018	0.38
V	46.25	1.32	0.016	0.37
VI	41.29	1.17	0.014	0.12
VII	35.75	1.04	0.013	0.15
VIII	30.56	0.87	0.010	0.17
IX	24.89	0.70	0.008	0.10
X	20.32	0.55	0.007	0.04

[a] For definition of parameters, see text.

situations are shown in Tables 7 and 8. A second order polynomial is enough for the steady state (Table 7), whereas fourth order is required for the cooling (Table 8). As can be expected from the low values that result for *sqdif*, the zero order constants (the *a* values in Table 7) not only fit the experimental value T_1 in the first column of Table 2 very precisely, but the full polynomial (eqn. (9a)) is quite good for the rest of the values of Table 2. This can be seen in Fig. 1, where the experimental data (asterisks) and the polynomial fit (continuous line) are practically indistinguishable.

The values of *sqdif* in Table 8, and the zero order constants a' being considerably higher than the experimental values T_1 in the first column of Table 2, lead to the conclusion that the fit to the cooling data is not as good as to the steady state. Cooling is not as easy to interpret as the steady state

TABLE 8

Parameters for the fit of experimental data for the cooling of the first point of the bar to a fourth order polynomial law (eqn. (9b))

Case selected	Parameter[a]					*sqdif*
	a'	$-b'$	c	$-d'$	e'	
I	76.68	34.54	6.55	0.57	0.019	24.5
III	60.62	27.25	5.14	0.45	0.015	23.5
VI	43.24	18.78	3.58	0.33	0.012	6.3
VIII	32.16	13.40	2.30	0.18	0.005	4.3

[a] For definition of parameters, see text.

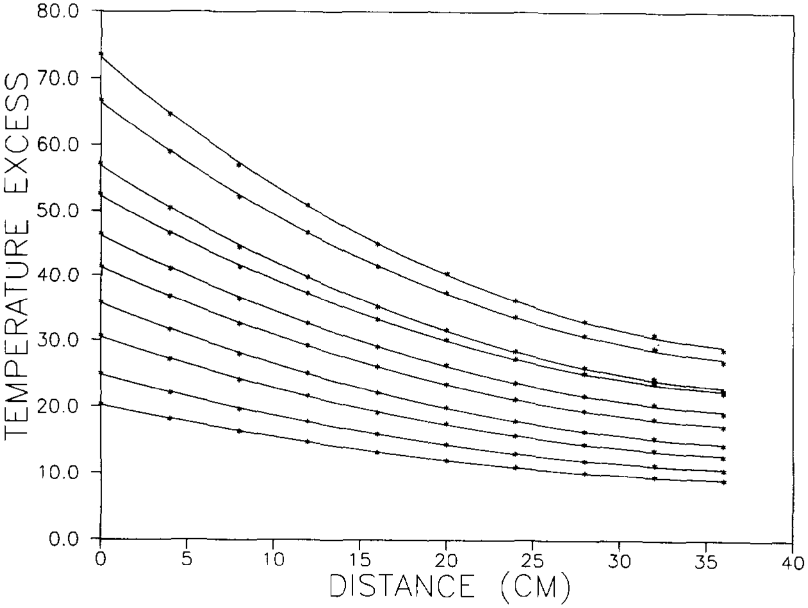

Fig. 1. Experimental temperature distributions for the steady states (*) compared with the fit to a second order polynomial (——).

because of the thermal inertia of a solid against changes in its temperature regime [9]. This can be seen in Fig. 2, where the first two or three points show the effect of this thermal inertia, which can be considered as a transitory regime before starting the cooling proper. These initial points

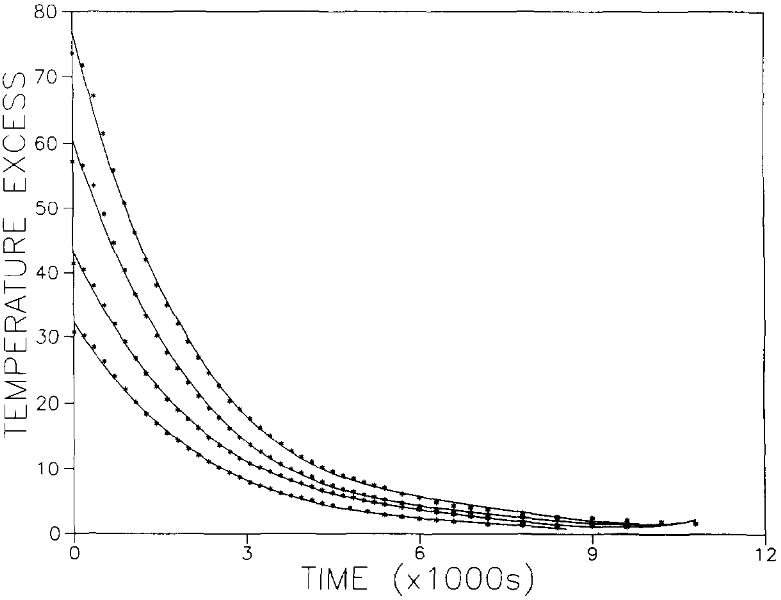

Fig. 2. As for Fig. 1, but for cooling and fitting with a fourth order polynomial.

give the main contribution to *sqdif*. For the remainder of the 40 experimental data points, the fit is just as good as in the steady state; one can see that experiment (asterisks) and polynomial fit (continuous line) coincide.

At this stage then a general polynomial expression for the steady state and cooling has been found which is the best governing equation for both temperature regimes. The next step will be to relate the polynomial coefficients with the thermal properties of the solid in the framework of a more general theory which requires further development.

DISCUSSION AND CONCLUSIONS

One frequently finds that elementary functions such as exponential, logarithmic, trigonometric or related functions do not explain the behaviour of some physical phenomenon satisfactorily because of the weak foundation of the approximations used in the theory. Perhaps the actual relationship between certain obviously related physical parameters is unknown, or perhaps the dependence of these parameters on time, position, etc. is very complex. In the present case of one-dimensional heat flow (eqn. (1)) Newton's law of cooling has been assumed to be valid, although this law was obtained for cooling of a body "in the draught" [2], i.e. by force convection, and with a small temperature excess over the surroundings so that all the physical properties which appear in eqn. (1) may be considered constant. These assumptions may not be valid for most experimental situations and a more elaborate theoretical treatment is necessary. The results here for an aluminium bar show that a more complex expression can be found to fit the data, using the hypergeometric functions [10] defined by the hypergeometric series

$$1+\frac{ab}{1!\,c}X+\frac{a(a+1)b(b+1)}{2!\,c(c+1)}X^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!\,c(c+1)(c+2)}X^3 \qquad (10)$$

The hypergeometric series is a generalization of the geometric series $1+X+X^2+X^3+\ldots$, and reduces to this when $a=1$ and $b=c$ in eqn. (10). It satisfies the hypergeometric equation, a second order linear differential equation with variable coefficients. Evidently, when the hypergeometric series is applied to the differential equation for heat flow in the bar, the coefficients a, b and c in eqn. (10) have to be related to those of eqn. (1) in some way.

Summarizing, if eqn. (1) is valid, and the steady state and cooling in a solid are fitted well by single exponentials such as eqns. (3a) and (5a) respectively, then one can obtain the value of the surface conductance H by using either eqn. (3b) or eqn. (5b), and as a further consequence, eqn. (6) is fully satisfied. If the best fit for both regimes is a double exponential, the values of H can also be determined directly by the appropriate combination of the two exponents into only one, giving an expression similar to eqns. (3b) and (5b). However, if the best fits are different from these functions, then a value for H cannot be obtained directly, because eqn. (1) is not satisfied, and a more elaborate theory must be developed to determine the surface conductance. The solution could be to use the hypergeometric functions, but this treatment requires further deeper studies with more experimental data for different materials.

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